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Goal-oriented error estimation for Stokes flow interacting with a flexible channel

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SUMMARY

We develop a goal-oriented error estimator for finite-element discretizations of fluid–structure-interaction problems. As a model problem, we consider the steady Stokes flow in a 2D channel where part of the channel wall is flexible. We introduce the reference domain approach where the Stokes problem on the variable domain is transformed into a fixed reference domain. This allows the formulation of a proper dual problem. The dual solution is then used in the evaluation of the error estimate, as usual. Copyright © 2007 John Wiley & Sons, Ltd.

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KEY WORDS: fluid–structure interaction; error estimation; adaptivity; duality; goal-oriented error estimation; free-boundary problems

1. INTRODUCTION

Numerical simulations of fluid–structure interaction typically require vast computational resources. Finite-element techniques employing goal-oriented adaptive strategies could offer a substantial improvement in the efficiency of such simulations. These strategies rely on *a posteriori* error estimates for specific output quantities of interest, the goal functionals. For this, an appropriate

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dual problem is required, see, for instance, [1, 2]. However, the free-boundary character of fluid–structure-interaction problems forms a fundamental complication, as it yields the underlying fluid domain unknown *a priori*. Consequently, the formulation of an appropriate dual problem is nontrivial.

In this work we develop a goal-oriented error estimator for finite-element discretizations of fluid–structure-interaction problems. As a model problem, we consider the steady Stokes flow in a 2D channel where part of the channel wall is flexible. To formulate an appropriate dual problem, we introduce the reference domain method. For this, we transfer the Stokes problem from the unknown domain to the fixed approximate domain using a transformation map. The dual solution is then used in the evaluation of the error estimate, as usual.

We note that this approach is different from [3], where it is proposed to embed the problem in a large enough hold-all domain prior to linearization. Furthermore, our framework does not involve a total Eulerian framework as considered in [4]. A goal-oriented error estimation for Stokes flow without interaction has been considered in, for instance, [5, 6].

2. PROBLEM STATEMENT

2.1. *Fluid–structure-interaction model*

We consider the fluid–structure system depicted in Figure 1. It corresponds to the half-length channel considered in [7]. For each (vertical) structure displacement $\alpha:\Gamma_0\to\mathbb{R}$, we associate the open bounded fluid domain Ω_α. It has a boundary $\partial\Omega_\alpha$ consisting of in- and outflow boundaries $\Gamma_{\text{in/out}}$, wall boundaries Γ_{wall} and the flexible segment Γ_α, which we refer to as the fluid–structure interface.

On the domain Ω_α, we consider the Stokes flow problem for the velocity $u:\Omega_\alpha\to\mathbb{R}^2$ and pressure $p:\Omega_\alpha\to\mathbb{R}$:

$$\left.\begin{aligned} -\Delta u+\nabla p&=0\\ -\nabla\cdot u&=0\end{aligned}\right\}\quad \text{in } \Omega_\alpha \tag{1a}$$

$$u=0 \quad \text{on } \Gamma_\alpha\cup\Gamma_{\text{wall}} \tag{1b}$$

$$u=u_{\text{in/out}} \quad \text{on } \Gamma_{\text{in/out}} \tag{1c}$$

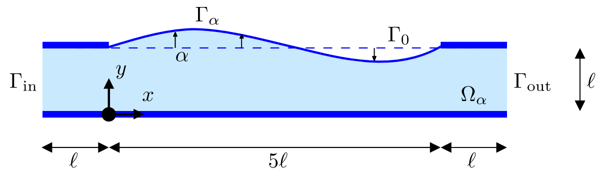

Figure 1. The fluid–structure system: its geometry definition; the fluid domain Ω_α and current fluid–structure interface Γ_α for vertical structure displacement α; the reference interface Γ_0 (dashed line); in- and outflow boundaries Γ_{in} and Γ_{out}. The thick lines correspond to the wall boundaries Γ_{wall}.

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The structure consists of a string constrained to move only vertically. Its displacement α, driven by the vertical component of the Stokes flow interface traction, satisfies

$$-T\alpha''(x)/\sqrt{1+\alpha'(x)^2}=(pn-\nabla u\cdot n)\cdot(0,1) \quad \text{on } \Gamma_\alpha \tag{1d}$$

$$\alpha=0 \quad \text{on } \partial\Gamma_\alpha \tag{1e}$$

where T is the string tension, which is assumed to be constant. Note that the displacement α maps a point on the reference interface, Γ_0, to a point on the current interface, Γ_α.

For the purpose of finite-element discretizations, we also consider the standard weak formulation of the Stokes problem:

$$\text{Find } (u,p)\in H^1_{u^{\mathrm{D}}}(\Omega_\alpha)\times L^2_0(\Omega_\alpha):$$

$$a(\alpha;u,v)+b(\alpha;v,p)=0 \quad \forall v\in H^1_0(\Omega_\alpha) \tag{2a}$$

$$b(\alpha;u,q)=0 \quad \forall q\in L^2_0(\Omega_\alpha) \tag{2b}$$

where

$$a(\alpha;u,v):=\int_{\Omega_\alpha}\nabla u:(\nabla v)^{\mathsf{T}}$$

$$b(\alpha;v,p):=-\int_{\Omega_\alpha}p\nabla\cdot v$$

Note that $H^1_{u^{\mathrm{D}}}(\Omega_\alpha)$ consists of functions satisfying the Dirichlet boundary conditions (1b) and (1c).

Assuming that the interface fluid traction is smooth enough, we have the following variational form for the structure:

$$\text{Find } \alpha\in H^1_0(\Gamma_0):$$

$$k(\alpha,\beta)=g(\alpha;u,p,\beta) \quad \forall\beta\in H^1_0(\Gamma_0) \tag{2c}$$

where

$$k(\alpha,\beta):=T\int_{\Gamma_0}\alpha'\beta'$$

$$g(\alpha;u,p,\beta):=\int_{\Gamma_\alpha}(pn-\nabla u\cdot n)\cdot(0,1)\beta(x)$$

2.2. Goal functionals

Our interest will be specific (bounded and differentiable) goal functionals $q:(\alpha,u,p)\in H^1_0(\Gamma_0)\times H^1_{u^{\mathrm{D}}}(\Omega_\alpha)\times L^2_0(\Omega_\alpha)\to\mathbb{R}$ of the solution, for example, the average displacement $\int_{\Gamma_0}\alpha$.

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Let $(\alpha^h, u^h, p^h) \in H_0^1(\Gamma_0) \times H_{\Gamma_D}^1(\Omega_{\alpha^h}) \times L_0^2(\Omega_{\alpha^h})$ be a Galerkin approximation of (2). Our objective is to provide an estimate for the goal error $\mathscr{E}_q := q(\alpha, u, p) - q(\alpha^h, u^h, p^h)$.

3. GOAL-ORIENTED ERROR ESTIMATION

3.1. Dual of the coupled problem

In generic boundary value problems, the dual problem involves the adjoint of the original primal problem. Since our problem is a free-boundary problem, it is nontrivial to obtain the appropriate dual. Therefore, we propose to first reformulate our problem to a fixed reference domain and subsequently perform a linearization to arrive at our dual problem.

Consider the dual variables z (dual to the velocity u), s (dual to the pressure p) and ζ (dual to the displacement α) with $\zeta = 0$ at $\partial\Gamma_0$. Multiplying the primal equations (1) with the dual variables and integrating over their corresponding domain, we obtain

$$\int_{\Omega_\alpha} (\nabla u : (\nabla z)^{\mathsf{T}} - p \nabla \cdot z) + \int_{\partial\Omega_\alpha} (pn - \nabla u \cdot n) \cdot z = 0 \tag{3a}$$

$$-\int_{\Omega_\alpha} s \nabla \cdot u = 0 \tag{3b}$$

$$\int_{\Gamma_0} T\alpha' \zeta' - \int_{\Gamma_\alpha} (pn - \nabla u \cdot n) \cdot (0,1)\zeta = 0 \tag{3c}$$

where we have performed an integration by parts in the first equation and last equation. Adding (3a)–(3c) together, the traction terms, $pn - \nabla u \cdot n$, on the interface Γ_α will vanish by choosing the following boundary condition for z:

$$z = \begin{cases} (0,1)\zeta & \text{on } \Gamma_\alpha \\ (0,0) & \text{on } \partial\Omega_\alpha \setminus \Gamma_\alpha \end{cases} \tag{4}$$

Before linearizing the functionals with respect to the displacement α around α^h, we introduce the transformation map $T_{\alpha^h,\alpha} : \Omega_{\alpha^h} \to \Omega_\alpha$ defined by

$$T_{\alpha^h,\alpha}(x,y) = \begin{cases} \left(x, \dfrac{\ell + \alpha(x)}{\ell + \alpha^h(x)} y\right), & x \in [0, 5\ell] \\ (x,y), & x \notin [0, 5\ell] \end{cases} \tag{5}$$

Applying this transformation yields the important identity

$$\hat{a}(\alpha; \hat{u}, \hat{z}) + \hat{b}(\alpha; \hat{z}, \hat{p}) + \hat{b}(\alpha; \hat{u}, \hat{s}) + k(\alpha, \zeta) = 0 \tag{6}$$

where we have introduced the transferred primal and dual Stokes flow unknowns

$$\hat{u} = u \circ T_{\alpha^h,\alpha}, \quad \hat{p} = p \circ T_{\alpha^h,\alpha}$$

$$\hat{z} = z \circ T_{\alpha^h,\alpha}, \quad \hat{s} = s \circ T_{\alpha^h,\alpha}$$

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and the following transferred forms:

$$\hat{a}(\alpha; \hat{u}, \hat{z}) := a(\alpha; \hat{u} \circ T^{-1}_{\alpha^h,\alpha}, \hat{z} \circ T^{-1}_{\alpha^h,\alpha}) = \int_{\Omega_{\alpha^h}} (A_{\alpha^h,\alpha} \cdot \hat{\nabla}\hat{u}) : (\hat{\nabla}\hat{z})^{\mathsf{T}}$$

$$\hat{b}(\alpha; \hat{u}, \hat{s}) := b(\alpha; \hat{u} \circ T^{-1}_{\alpha^h,\alpha}, \hat{s} \circ T^{-1}_{\alpha^h,\alpha}) = -\int_{\Omega_{\alpha^h}} \hat{s}(B_{\alpha^h,\alpha} \cdot \hat{\nabla}) \cdot \hat{u}$$

with $\hat{\nabla}$ denoting differentiation in the reference domain Ω_{α^h}. The terms appearing due to the transformation are

$$A_{\alpha^h,\alpha} := J_{\alpha^h,\alpha} DT^{-1}_{\alpha^h,\alpha} \cdot DT^{-\mathsf{T}}_{\alpha^h,\alpha}$$

$$B_{\alpha^h,\alpha} := J_{\alpha^h,\alpha} DT^{-\mathsf{T}}_{\alpha^h,\alpha}$$

$$J_{\alpha^h,\alpha} := \det DT_{\alpha^h,\alpha}$$

where $DT_{\alpha^h,\alpha} : \Omega_{\alpha^h} \to \mathbb{R}^{2\times 2}$ is the Jacobian matrix of the transformation $T_{\alpha^h,\alpha}$ introduced in (5).

Applying this transformation to the goal functional gives the transferred form

$$\hat{q}(\alpha, \hat{u}, \hat{p}) = q(\alpha, \hat{u} \circ T^{-1}_{\alpha^h,\alpha}, \hat{p} \circ T^{-1}_{\alpha^h,\alpha})$$

Note that we have transformed our problem and functional into the fixed reference domain Ω_{α^h}; we can easily derive our dual by linearization:

Find $(\zeta, z, s) \in H^1_0(\Gamma_0) \times H^1_{z\mathrm{D}}(\Omega_{\alpha^h}) \times L^2_0(\Omega_{\alpha^h})$:

$$\hat{a}(\alpha^h; \delta u, z) + \hat{b}(\alpha^h; \delta u, s) = \partial_u \hat{q}(\alpha^h, u^h, p^h)(\delta u) \quad \forall \delta u \in H^1_0(\Omega_{\alpha^h}) \tag{7a}$$

$$\hat{b}(\alpha^h; z, \delta p) = \partial_p \hat{q}(\alpha^h, u^h, p^h)(\delta p) \quad \forall \delta p \in L^2_0(\Omega_{\alpha^h}) \tag{7b}$$

$$\partial_\alpha \hat{a}(\alpha^h; u^h, z)(\delta\alpha) + \partial_\alpha \hat{b}(\alpha^h; u^h, s)(\delta\alpha) + \partial_\alpha \hat{b}(\alpha^h; z, p^h)(\delta\alpha) + k(\delta\alpha, \zeta) = \partial_\alpha \hat{q}(\alpha^h, u^h, p^h)(\delta\alpha) \quad \forall \delta\alpha \in H^1_0(\Gamma_0) \tag{7c}$$

where $H^1_{z\mathrm{D}}(\Omega_{\alpha^h})$ consists of functions in $H^1(\Omega_{\alpha^h})$ satisfying boundary condition (4). Note that we need only the transformed functionals to perform the linearization with respect to the displacement α, yielding contributions

$$\partial_\alpha \hat{a}(\alpha^h; u^h, z)(\delta\alpha) = \int_{\Omega_{\alpha^h}} (\langle \partial_\alpha A_{\alpha^h,\alpha}, \delta\alpha \rangle \cdot \hat{\nabla} u^h) : (\hat{\nabla} z)^{\mathsf{T}}$$

$$\partial_\alpha \hat{b}(\alpha^h; z, p^h)(\delta\alpha) = -\int_{\Omega_{\alpha^h}} p^h (\langle \partial_\alpha B_{\alpha^h,\alpha}, \delta\alpha \rangle \cdot \hat{\nabla}) \cdot z$$

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3.2. *A posteriori error estimate*

Theorem 1
Given any approximation $(\alpha^h, u^h, p^h) \in H_0^1(\Gamma_0) \times H^1_{u^{\mathrm{D}}}(\Omega_{\alpha^h}) \times L_0^2(\Omega_{\alpha^h})$ to the primal problem (1), let $(\zeta, z, s) \in H_0^1(\Gamma_0) \times H^1_{z^{\mathrm{D}}}(\Omega_{\alpha^h}) \times L_0^2(\Omega_{\alpha^h})$ be the solution to the dual problem (7). Then, we have the error representation

$$\begin{aligned}\mathscr{E}_q &= q(\alpha, u, p) - q(\alpha^h, u^h, p^h) \\ &= -(a(\alpha^h, u^h, z) + b(\alpha^h, z, p^h) + b(\alpha^h, u^h, s) + k(\alpha^h, \zeta)) + \mathscr{O}(\|e\|^2)\end{aligned}$$

where $\|e\| = \|e^\alpha\|_{H_0^1(\Gamma_0)} + \|e^u\|_{H_0^1(\Omega_{\alpha^h})} + \|e^p\|_{L_0^2(\Omega_{\alpha^h})}$ and the errors are defined as

$$e^\alpha := \alpha - \alpha^h, \quad e^u := u \circ T_{\alpha^h,\alpha} - u^h, \quad e^p := p \circ T_{\alpha^h,\alpha} - p^h$$

Note that the residual $\mathscr{R}_q := -(a(\alpha^h, u^h, z) + b(\alpha^h, z, p^h) + b(\alpha^h, u^h, s) + k(\alpha^h, \zeta))$ serves as the error estimate and can be approximated by computing a discrete dual solution.

Proof of Theorem 1
First note that

$$\mathscr{E}_q = \hat{q}(\alpha, u \circ T_{\alpha^h,\alpha}, p \circ T_{\alpha^h,\alpha}) - \hat{q}(\alpha^h, u^h, p^h) = \hat{q}'(\alpha^h, u^h, p^h)(e^\alpha, e^u, e^p) + \mathscr{O}(\|e\|^2)$$

Then using the dual problem (7), we have

$$\begin{aligned}\mathscr{E}_q &= \hat{a}(\alpha^h; e^u, z) + \hat{b}(\alpha^h; e^u, s) + \hat{b}(\alpha^h; z, e^p) + k(e^\alpha, \zeta) \\ &\quad + \partial_\alpha \hat{a}(\alpha^h; u^h, z)(e^\alpha) + \partial_\alpha \hat{b}(\alpha^h; u^h, s)(e^\alpha) + \partial_\alpha \hat{b}(\alpha^h; z, p^h)(e^\alpha) + \mathscr{O}(\|e\|^2)\end{aligned}$$

and since the dual was a linearization of the left-hand side of (6), we obtain

$$\begin{aligned}\mathscr{E}_q &= (\hat{a}(\alpha; u \circ T_{\alpha^h,\alpha}, z) + \hat{b}(\alpha; u \circ T_{\alpha^h,\alpha}, s) + \hat{b}(\alpha; z, p \circ T_{\alpha^h,\alpha}) + k(\alpha, \zeta)) \\ &\quad - (\hat{a}(\alpha^h; u^h, z) + \hat{b}(\alpha^h; u^h, s) + \hat{b}(\alpha^h; z, p^h) + k(\alpha^h, \zeta)) + \mathscr{O}(\|e\|^2)\end{aligned}$$

Finally, we obtain the proof by noting that the first term in brackets vanishes due to identity (6). □

4. NUMERICAL EXPERIMENT

We consider our primal problem with parameters $\ell = 1$ and $T = 50$, and $u_{\text{in/out}}$ corresponds to a parabolic in- and outflow with a maximum of $\frac{3}{2}$. We discretize the Stokes flow using the standard $(P2 - P1)$ Taylor–Hood finite element on triangles. The structure equation is solved using linear finite elements, for which its elements exactly correspond to the adjacent fluid–element edges. The coupled problem is solved using subiteration, where at each iteration the fluid mesh is deformed

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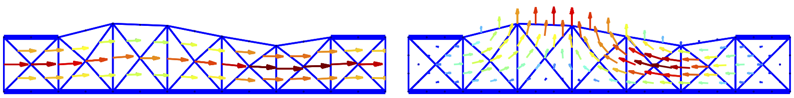

Figure 2. The primal (left) and dual (right) solution on a 28-element mesh. The arrows represent the velocity vectors u^h and z. Note that ζ is also readable from the figure since $z=(0,1)\zeta$ at the interface Γ_{α^h}.

to accommodate the new displacement. The dual problem is discretized using the same mesh as the primal problem, but the dual shape functions are of one order higher (to prevent Galerkin orthogonality that would yield a zero error estimate). The goal functional is the average structure displacement $\int_{\Gamma_0} \alpha$.

Figure 2 shows a sample primal and dual solution obtained on a mesh of 28 elements. In the table below, we report the convergence of the estimator on uniform meshes demonstrating the consistency of our goal-oriented error estimator.

	Elements	Dofs	$q=\int_{\Gamma_0}\alpha^h$	Error $\mathscr{E}_q$	Estimate $\mathscr{R}_q$	Effectivity $\mathscr{R}_q/\mathscr{E}_q$
	28	176	0.32261	0.06044	0.0815414	1.35
	112	599	0.37018	0.01287	0.0188627	1.47
	448	2201	0.38006	0.00300	0.0045154	1.51
	1792	8429	0.38230	0.00075	0.0010786	1.44
Reference	7168	32981	0.38305	0		

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